Rutgers University: Algebra Written Qualifying Exam

January 2016: Problem 1 Solution

Exercise. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, and assume that $P(0)$ and $P(1)$ are odd integers. Prove that $P(x)$ has no integer roots.

Solution.

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \quad a_i \in \mathbb{Z}$$

$$P(0) = a_0 \text{ is odd} \qquad \text{and} \quad P(1) = a_n + \cdots + a_1 + a_0 \text{ is odd}$$

$$\implies P(1) - P(0) = a_n + \cdots + a_1 \text{ is even}$$

$$\implies \text{an even number of } a_i \in \{a_n, \dots, a_1\} \text{ are odd}$$

$$m \in \mathbb{Z} \text{ is a root of } P(x) \iff P(m) = 0$$

If $m \in \mathbb{Z}$ is even, then $P(m) = a_n m^n + \cdots + a_1 m + a_0 = m(a_n + \cdots + a_1) + a_0$ is odd

$\implies P(m) \neq 0$

If $m \in \mathbb{Z}$ is odd, then $a_i m^i$ is odd IFF a_i is odd.

But there is an even number of $a_i \in \{a_n, \dots, a_1\}$ s.t. a_i is odd.

$\implies$ There is an even number of $a_i m^i \in \{a_n m^n, \dots, a_1 m\}$ s.t. $a_i m^i$ is odd.

$\implies a_n m^n + \cdots + a_1 m$ is even

$\implies P(m) = a_n m^n + \cdots + a_1 m + a_0$ is odd

$\implies P(m) \neq 0$

Thus, $P(x)$ has no integer roots.